Derivations of Gel and Nickel-Plated Wire Supercooled Liquid Water Content (SLWC) Equations

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Expanded and adapted from previous internal papers.

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Introduction

Anasphere's vibrating-wire sonde records a vibrating wire's frequency as ice collects along its length. These frequency measurements, combined with collocated meteorological measurements, can be used to determine the supercooled liquidwater content (SLWC) in the surrounding air. The following is a derivation of the equations used to calculate SLWC, for the gel-coated and nickel-plated wires, using the sonde's measurements.

Equation Derivation for Nickel Plated and Gel-Coated Wires

The sonde measures the frequency of the wire. This frequency is used to calculate the water collected by the wire, in the form of ice or absorbed liquid, by calculating the change in mass of the wire. Therefore, the actual frequency is not so important as is the change in frequency, Δf, which intern yields the change in mass, ΔM.

The gel-coated and Ni-plated wires differ from the usual, plain wire versions, in that there is a step in the solid structure of the geometry (Figure 1). This step is accounted for, by adding another term to the equations of motion, and integrating.

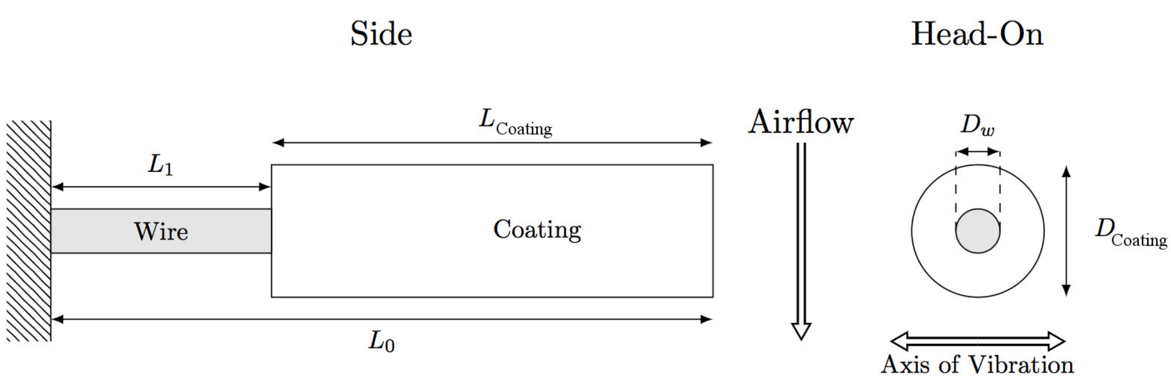

Figure 1: Plated or Coated Wire

The Rayleigh method is used to calculate the system frequency, by relating the potential and kinetic energies:

$$\omega^2 = \frac{\omega^2 PE_{max}}{KE_{max}}. \tag{1}$$

The displacement equation for the wire is assumed to be

$$y = B\cos(\omega t)\left(\cos\left(\frac{\pi x}{2L_0}\right) - 1\right), \tag{2}$$

where B is an un-specified amplitude and, recalling ω=2πf, f is the wire's vibration frequency.

The potential energy of the wire then, incorporating (2), is given by

$$PE = \frac{E_w I_w}{2}\int_0^{L_0} {y^2}_{xx} dx + \frac{E_{Cr} I_{Cr}}{2}\int_{L_1}^{L_0} {y^2}_{xx} dx + \frac{E_g I_g}{2}\int_{L_1}^{L_0} {y^2}_{xx} dx\,, \tag{3}$$

$$= \left(\frac{\pi^4 B^2}{64{L_0}^4}\right)\left(E_w I_w L_0 \cos^2(\omega t) + E_{Cr} I_{Cr} c L_0 \cos^2(\omega t) + E_g I_g c L_0 \cos^2(\omega t)\right),$$

$$= \left(\frac{\pi^4 B^2}{64{L_0}^4}\right) L_0 \cos^2(\omega t)\left(E_w I_w + E_{Cr} I_{Cr} c + E_g I_g c\right),$$

where y_{xx} is the second derivative of y with respect to x, E is the Young's modulus, and I is the area-moment of inertia. Here, the w subscript indicates the wire, the Cr subscript indicates the ceramic substrate, and the g subscript indicates the gel. Also, nickel and ice are represented by Ni, and i subscripts, respectively.

The area-moments of inertia are

$$I_w = \frac{\pi}{64}{D_w}^4\,, \tag{4}$$

$$I_{Cr} = \frac{\pi}{64}\left({D_{Cr}}^4 - {D_w}^4\right), \tag{5}$$

$$I_g = \frac{\pi}{64}\left({D_g}^4 - {D_w}^4\right),$$

$$I_{Ni} = \frac{\pi}{64}\left({D_{Ni}}^4 - {D_w}^4\right),$$

where $D_{Cr} = D_g = D_{Ni} = D_{Coating}$ (see Figure 1). Because the ceramic substrate absorbs the gel, the diameter terms, $D_{ceramic}$ and D_{gel}, are assumed to be nominally equal. For ice,

$$I_i = \frac{\pi}{12}\alpha {D_w}^4\,. \tag{6}$$

The kinetic energy of the wire is

$$KE = \frac{\lambda_w}{2}\int_0^{L_0} \dot{y}^2 dx + \frac{\lambda_{Cr}}{2}\int_{L_1}^{L_0} \dot{y}^2 dx + \frac{\lambda_g}{2}\int_{L_1}^{L_0} \dot{y}^2 dx\,, \tag{7}$$

$$= \frac{1}{2}\omega^2 B^2 L_0 \sin^2(\omega t)\left(a\lambda_w + b\lambda_{Cr} + b\lambda_g\right),$$

where λ_w, λ_{Cr}, and λ_g are the linear densities of the wire, ceramic substrate and gel, respectively. The constants a, b and c are given by

$$a = \frac{3\pi - 8}{2\pi} \tag{8}$$

$$b = a - \frac{3L_1}{2L_0} - \frac{1}{2\pi}\sin\left(\frac{\pi L_1}{2L_0}\right) + \frac{4}{\pi}\sin\left(\frac{\pi L_1}{2L_0}\right)$$

$$c = 1 - \frac{L_1}{L_0} - \frac{1}{\pi}\sin\left(\frac{\pi L_1}{L_0}\right).$$

Setting the sine and cosine terms in (3) and (7) to one, substituting them into (1), along with the relation $\omega = 2\pi f$, yields

$$f^2 = \frac{\pi^2\left(E_w I_w + E_{Cr} I_{Cr} c_c + E_g I_g c_c\right)}{128{L_0}^4\left(a_c \lambda_w + b_c \lambda_{Cr} + b_c \lambda_g\right)}, \tag{9}$$

where f is frequency. This equation can be rearranged to find the unknown linear density, which in this case, is for the gel, λ_g:

$$\lambda_g = \frac{\left(\frac{\pi^2(E_w I_w + E_{Cr} I_{Cr} c)}{128 f^2 {L_0}^4}\right) - a\lambda_w - b\lambda_{Cr}}{b}. \tag{10}$$

The change in mass, ΔM_i, is found by multiplying the change in linear density, from t_0 to t_1, by the length of the wire, L_0 (i.e. from f_0 to f_1 in (10)):

$$\Delta M_g = L_0\left(\lambda_{g1} - \lambda_{g0}\right), \tag{11}$$

where t_0 is the beginning, or pre-measurement time, and t_1 is the time at some point of interest in the sample data. As such, the frequency at t_0 would be the beginning pre-measurement, or base frequency, f_0, and at t_1, would be the frequency at some point of interest in the sample data, f_1, etc...

The same method applies to the nickel-plated wire. ΔM_i, is calculated by replacing all ceramic substrate terms with the corresponding terms for the nickel, and replacing all gel terms with the corresponding terms for the ice. These substitutions must be made throughout, but as one example, (11) becomes

$$\Delta M_i = L_0(\lambda_{i1} - \lambda_{i0}), \tag{12}$$

where ΔM_i is the change in mass of the ice, and λ_{i0} and λ_{i1} are the linear mass densities at f_0 and f_1, respectively.

The supercooled liquid water content (SLWC) can then be calculated by substituting the result of (11) or (12) into the equation

$$SLWC = \frac{\Delta M_g}{\epsilon V} = \frac{\Delta M_g}{\epsilon D_w L_i v \Delta t}. \tag{13}$$

This is based on the model used in Hill (1994). An equation for SLWC is found by starting with the basic equation for a length of wire sweeping through a volume V of air, with speed v, and collecting a percentage (collection efficiency) ε of the SLW, contained in that volume, as ice on its forward edge, over time, t.

The collection efficiency of the wire is calculated using the method of Lozowski et al. (1983):

$$\epsilon = \begin{Bmatrix} 0.489(\log_{10}(8K_0))^{1.978} & 0.125 \le K_0 < 0.9 \\ \dfrac{K_0}{\frac{\pi}{2} + K_0} & 0.9 \le K_0 \end{Bmatrix}, \tag{14}$$

$$K_0 = 0.125 + \frac{K - 0.125}{1 + 0.0967 Re^{0.6367}},$$

$$K = \frac{\rho_w v d^2}{9 \mu_a D},$$

$$Re = \frac{d v \rho_a}{\mu_a}.$$

Here, K_0 is the modified Langmuir inertia parameter, K is the Langmuir inertia parameter, Re is the Reynolds number, ρ_w is the density of water, d is the droplet diameter, ρ_a is the air density and μ_a is the air viscosity. In practice, the efficiency is essentially zero for $K_0 < 0.125$. D is the diameter of the cylinder collecting water, which in the case of the icing sensor would be D_w.

The air viscosity is calculated using Sutherland's Law:

$$\mu_a = \mu_0 \frac{T_0 + S}{T + S} \left(\frac{T}{T_0}\right)^{3/2}, \tag{15}$$

where T is the air temperature, T_0 is a reference temperature, μ_0 is the viscosity at T_0, and S is Sutherland's temperature for air. Some convenient values are

$$T_0 = 273.15\ K,$$

$$\mu_0 = 1.716\times10^{-5} \frac{kg}{ms},$$

$$S = 110.4\ K.$$

References

Hill, G. "Analysis of Supercooled Liquid Water Measurements Using Microwave Radiometer and Vibrating Wire Devices." J. Atmos. Ocean. Technol. **11**, 1242-1252 (1994).

"Supercooled Liquid Water Content (SLWC) Sensor Equations.", Fred Bunt, Anasphere, Inc. (2017).

"Supercooled Liquid Water Content (SLWC) Equation Derivation.", Fred Bunt, Anasphere, Inc. (2017).

Appendices

A Energy Integrals

The assumed displacement equation, as referenced above, repeated here:

$$y = B\cos(\omega t)\left(\cos\left(\frac{\pi x}{2L_0}\right) - 1\right) \tag{2}$$

A.1 Derivatives

The derivatives of the displacement equation, equation 2:

$$\dot{y} = -\omega B\sin(\omega t)\left[\cos\left(\frac{\pi x}{2L_0}\right) - 1\right] \tag{17}$$

$$\dot{y}^2 = \omega^2 B^2 \sin^2(\omega t)\left[\cos^2\left(\frac{\pi x}{2L_0}\right) - 2\cos\left(\frac{\pi x}{2L_0}\right) + 1\right] \tag{18}$$

$$y_x = -\left(\frac{\pi B}{2L_0}\right)\cos(\omega t)\sin\left(\frac{\pi x}{2L_0}\right) \tag{19}$$

$$y_{xx} = -\left(\frac{\pi^2 B}{4{L_0}^2}\right)\cos(\omega t)\cos\left(\frac{\pi x}{2L_0}\right) \tag{20}$$

$${y_{xx}}^2 = -\left(\frac{\pi^4 B^2}{16{L_0}^4}\right)\cos^2(\omega t)\cos^2\left(\frac{\pi x}{2L_0}\right) \tag{21}$$

A.2 Integrals

A.2.1 Time-Derivative Integrals

Integrate the derivatives of the assumed displacement equation, to be used in deriving the kinetic energy equation (7) above:

$$\int \dot{y}^2 dx = \omega^2 B^2 \sin^2(\omega t)\int\left[\cos^2\left(\frac{\pi x}{2L_0}\right) - 2\cos\left(\frac{\pi x}{2L_0}\right) + 1\right]dx$$

$$= \omega^2 B^2 \sin^2(\omega t)\left[\frac{1}{2}\int\left(1 + \cos\left(\frac{\pi x}{L_0}\right)\right)dx - \frac{4L_0}{\pi}\sin\left(\frac{\pi x}{2L_0}\right) + x\right]$$

$$= \omega^2 B^2 \sin^2(\omega t)\left[\frac{3}{2}x + \frac{L_0}{2\pi}\sin\left(\frac{\pi x}{L_0}\right) - \frac{4L_0}{\pi}\sin\left(\frac{\pi x}{2L_0}\right)\right]$$

$$\int_0^{L_0} \dot{y}^2\, dx = \omega^2 B^2 \sin^2(\omega t)\left[\frac{3}{2}x + \frac{L_0}{2\pi}\sin\left(\frac{\pi x}{L_0}\right) - \frac{4L_0}{\pi}\sin\left(\frac{\pi x}{2L_0}\right)\right|_{0}^{L_0}$$

$$= \omega^2 B^2 \sin^2(\omega t)\left[\frac{3}{2}L_0 + \frac{L_0}{2\pi}(\sin(\pi) - \sin(0)) - \frac{4L_0}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right)\right]$$

$$= \omega^2 B^2 \sin^2(\omega t)\left[\frac{3}{2}L_0 - \frac{4}{\pi}L_0\right]$$

$$= \omega^2 B^2 a\, L_0 \sin^2(\omega t) \tag{22}$$

Find constant a:

$$a = \frac{3}{2} - \frac{4}{\pi} = \frac{3\pi - 8}{2\pi} \tag{8a}$$

$$\int_{L_1}^{L_0} \dot{y}^2\, dx = \omega^2 B^2 \sin^2(\omega t)\left[\frac{3}{2}x + \frac{L_0}{2\pi}\sin\left(\frac{\pi x}{L_0}\right) - \frac{4L_0}{\pi}\sin\left(\frac{\pi x}{2L_0}\right)\right|_{L_1}^{L_0}$$

$$= \omega^2 B^2 \sin^2(\omega t)\left[\frac{3}{2}(L_0 - L_1) + \frac{L_0}{2\pi}\left(\sin(\pi) - \sin\left(\frac{\pi L_1}{L_0}\right)\right) - \frac{4L_0}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi L_1}{2L_0}\right)\right)\right]$$

$$= \omega^2 B^2 \sin^2(\omega t)\left[\frac{3}{2}(L_0 - L_1) + \frac{L_0}{2\pi}\sin\left(\frac{\pi L_1}{L_0}\right) - \frac{4L_0}{\pi} + \frac{4L_0}{\pi}\sin\left(\frac{\pi L_1}{2L_0}\right)\right]$$

$$= \omega^2 B^2 \sin^2(\omega t)\left[\left(\frac{3}{2} - \frac{4}{\pi}\right)L_0 - \frac{3}{2}L_1 - \frac{L_0}{2\pi}\sin\left(\frac{\pi L_1}{L_0}\right) + \frac{4L_0}{\pi}\sin\left(\frac{\pi L_1}{2L_0}\right)\right]$$

$$= \omega^2 B^2\, \mathrm{b} L_0 \sin^2(\omega t) \tag{23}$$

Find constant b:

$$b = a - \frac{3L_1}{2L_0} - \frac{1}{2\pi}\sin\left(\frac{\pi L_1}{L_0}\right) + \frac{4}{\pi}\sin\left(\frac{\pi L_1}{2L_0}\right) \tag{8b}$$

A.2.2 X-Derivative Integrals

Integrate the derivatives of the assumed displacement equation, to be used in the derivation of the potential energy equation (3) above:

$$\int {y_{xx}}^2\,dx = \left(\frac{\pi^4 B^2}{16{L_0}^4}\right)\cos^2(\omega t)\int \cos^2\left(\frac{\pi x}{2L_0}\right)dx$$

$$= \frac{1}{2}\left(\frac{\pi^4 B^2}{16{L_0}^4}\right)\cos^2(\omega t)\int \left(1+\cos\left(\frac{\pi x}{L_0}\right)\right)dx$$

$$= \left(\frac{\pi^4 B^2}{32{L_0}^4}\right)\cos^2(\omega t)\left[x+\frac{L_0}{\pi}\sin\left(\frac{\pi x}{L_0}\right)\right]$$

$$\int_0^{L_0} {y_{xx}}^2\,dx = \left(\frac{\pi^4 B^2}{32{L_0}^4}\right)\cos^2(\omega t)\left[x+\frac{L_0}{\pi}\sin\left(\frac{\pi x}{L_0}\right)\right]_0^{L_0}$$

$$= \left(\frac{\pi^4 B^2}{32{L_0}^4}\right)L_0\cos^2(\omega t) \tag{24}$$

$$\int_{L_1}^{L_0} {y_{xx}}^2\,dx = \left(\frac{\pi^4 B^2}{32{L_0}^4}\right)\cos^2(\omega t)\left[x+\frac{L_0}{\pi}\sin\left(\frac{\pi x}{L_0}\right)\right]_{L_1}^{L_0}$$

$$= \left(\frac{\pi^4 B^2}{32{L_0}^4}\right)\cos^2(\omega t)\left[L_0 - L_1 + \frac{L_0}{\pi}\sin(\pi) - \frac{L_0}{\pi}\sin\left(\frac{\pi L_1}{L_0}\right)\right]$$

$$= \left(\frac{\pi^4 B^2}{32{L_0}^4}\right)L_0\cos^2(\omega t)\left[1-\frac{L_1}{L_0}-\frac{1}{\pi}\sin\left(\frac{\pi L_1}{L_0}\right)\right]$$

$$= \left(\frac{\pi^4 B^2}{32{L_0}^4}\right)L_0\cos^2(\omega t)\left[1-\frac{L_1}{L_0}-\frac{1}{\pi}\sin\left(\frac{\pi L_1}{L_0}\right)\right]$$

$$= \left(\frac{\pi^4 B^2}{32{L_0}^4}\right)cL_0\cos^2(\omega t) \tag{25}$$

Find constant c:

$$c = 1-\frac{L_1}{L_0}-\frac{1}{\pi}\sin\left(\frac{\pi L_1}{L_0}\right) \tag{8c}$$