Supercooled Liquid Water Content (SLWC) Sensor Equations

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SLWC Equations.

The following discussion pertains to Anasphere's supercooled liquid water content (SLWC) sensor, which measures the supercooled liquid water content of clouds. Key user-provided parameters include the median volume diameter of droplets, air velocity, and ambient meteorological parameters.

A separate discussion pertains to the related liquid water content (LWC) sensor; do not use the following math for that sensor.

Very important: it is recommended that the raw frequency data be smoothed before applying the following equations. We do not recommend using a moving average. Rather, applying a Savitzky–Golay filter with a window of nominally 15 points (subject to case-by-case optimization) is recommended to eliminate higher frequency noise.

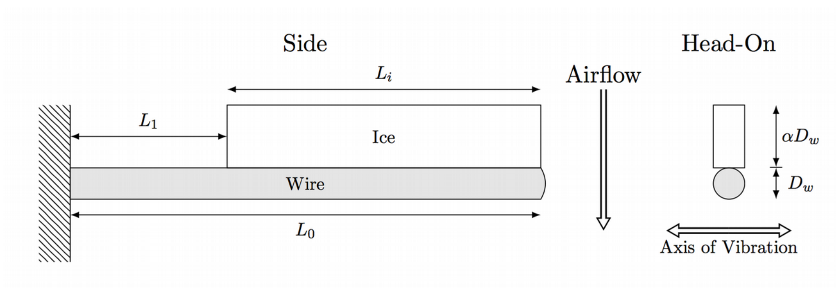

Figure 1. SLWC wire loaded with ice.

The SLWC sensor is modeled as shown in Figure 1. Here, α is a scalar allowing the ice's height to be expressed in terms of D_w. This is the same model used in Hill (1994). An equation for SLWC is found by starting with the basic equation for a length of wire sweeping through a volume V of air, with speed v, and collecting a percentage (collection efficiency) ε of the SLW contained in that volume as ice on its forward edge:

$$SLWC = \frac{\Delta M_i}{\epsilon V} = \frac{\Delta M_i}{\epsilon D_w L_i v \Delta t}. \tag{1}$$

M_i is the accumulated ice mass and t is time. Turning the mass into a linear density λ_i along the accumulation length and reducing the changing terms into infinitesimals, this becomes

$$\begin{aligned} SLWC &= \frac{\Delta \lambda_i}{\epsilon D_w v \Delta t}, \\ &= \frac{1}{\epsilon D_w v} \frac{d\lambda_i}{dt}, \\ &= \frac{1}{\epsilon D_w v} \dot{\lambda}_i. \end{aligned} \tag{2}$$

Next, the linear density needs to be expressed in terms of the wire's frequency f and then differentiated with respect to time. To do this, the Rayleigh method is used:

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$$\omega^2 = \frac{\omega^2 PE_{max}}{KE_{max}}, \tag{3}$$

where $\omega=2\pi f$ and PE_{max} and KE_{max} are the maximum potential and kinetic energy of the wire, respectively. The displacement equation for the wire is assumed to be

$$y = B\cos(\omega t)\left(\cos\left(\frac{\pi x}{2L_0}\right) - 1\right), \tag{4}$$

where B is an unspecified amplitude. The potential energy of the wire is then given by

$$\begin{aligned} PE &= \frac{E_w I_w}{2}\int_0^{L_0} y_{xx}^2\, dx + \frac{E_i I_i}{2}\int_{L_1}^{L_0} y_{xx}^2\, dx, \\ &= \left(\frac{\pi^4 B^2}{64 L_0^4}\right) E_w L_0 \cos^2(\omega t)\left(I_w + \frac{cI_i}{\beta}\right), \end{aligned} \tag{5}$$

where y_{xx} is the second derivative of y with respect to x, E is the Young's modulus, I is the area-moment of inertia, and $\beta=E_w/E_i$. Here, the w subscript indicates the wire and the i subscript indicates the ice. The area-moments of inertia are

$$I_w = \frac{\pi}{64} D_w^4, \tag{6}$$

$$I_i = \frac{1}{12}\alpha D_w^4. \tag{7}$$

The kinetic energy of the wire is

$$\begin{aligned} KE &= \frac{\lambda_w}{2}\int_0^{L_0} \dot{y}^2\, dx + \frac{\lambda_i}{2}\int_{L_1}^{L_0} \dot{y}^2\, dx, \\ &= \frac{1}{2}\omega^2 B^2 L_0 \sin^2(\omega t)(a\lambda_w + b\lambda_i), \end{aligned} \tag{8}$$

where λ_w is the linear density of the wire. The constants a, b and c are given by

$$\begin{aligned} a &= \frac{3\pi - 8}{2\pi} \\ b &= a - \frac{3L_1}{2L_0} - \frac{1}{2\pi}\sin\left(\frac{\pi L_1}{L_0}\right) + \frac{4}{\pi}\sin\left(\frac{\pi L_1}{2L_0}\right) \\ c &= 1 - \frac{L_1}{L_0} - \frac{1}{\pi}\sin\left(\frac{\pi L_1}{L_0}\right). \end{aligned} \tag{9}$$

Setting the sine and cosine terms in (5) and (8) to one and using them with (3), and recalling that $\omega=2\pi f$, yields

$$f^2 = \frac{\pi^2 E_w\left(I_w + \frac{cI_i}{\beta}\right)}{128 L_0^4 (a\lambda_w + b\lambda_i)}. \tag{10}$$

To find $\lambda_i(f)$, I_i must first be written in terms of the ice's linear density. This is done by noting that

$$\lambda_i = \rho_i A_i = \rho_i \alpha D_w^2, \tag{11}$$

where ρ_i is the ice's density and A_i is the ice's cross-sectional area, then solving for α,

$$\alpha = \frac{\lambda_i}{\rho_i D_w^2}, \tag{12}$$

and using the result in (7). I_i becomes

$$I_i = \frac{D_w^2}{12\rho_i}\lambda_i = \phi\lambda_i, \tag{13}$$

where

$$\phi = \frac{D_w^2}{12\rho_i}. \tag{14}$$

Thus, (10) becomes

$$f^2 = \frac{a f_0^2 \lambda_w \left(I_w + \frac{c\phi}{\beta}\lambda_i\right)}{I_w(a\lambda_w + b\lambda_i)}, \tag{15}$$

where f_0 is the wire's fundamental frequency:

$$\begin{aligned} f_0^2 &= f^2(\lambda_i = 0) \\ &= \frac{\pi^2 E_w I_w}{128 L_0^4 a \lambda_w}. \end{aligned} \tag{16}$$

From here, through some algebra, the ice's linear density can be written as a function of the wire's frequency:

$$\lambda_i = \frac{b_0 I_w\left(\frac{f^2}{f_0^2} - 1\right)}{\frac{b_0 c\phi}{\beta} - \frac{f^2}{f_0^2} I_w} \tag{17}$$

$$b_0 = \frac{a}{b}\lambda_w. \tag{18}$$

The time-derivative of the ice's linear density is then

$$\dot{\lambda}_i = \frac{2 b_0 \beta I_w f_0^2 f (b_0 c\phi - \beta I_w)}{(b_0 c\phi f_0^2 - \beta I_w f^2)^2}\dot{f}. \tag{19}$$

Using this in (2) produces

$$SLWC = \frac{2 b_0 \beta I_w f_0^2 f (b_0 c\phi - \beta I_w)}{\epsilon D v (b_0 c\phi f_0^2 - \beta I_w f^2)^2}\dot{f} \tag{20}$$

as the final equation for SLWC, where

$$a=\frac{3\pi-8}{2\pi},$$

$$b=a-\frac{3L_1}{2L_0}-\frac{1}{2\pi}\sin\left(\frac{\pi L_1}{L_0}\right)+\frac{4}{\pi}\sin\left(\frac{\pi L_1}{2L_0}\right),$$

$$c=1-\frac{L_1}{L_0}-\frac{1}{\pi}\sin\left(\frac{\pi L_1}{L_0}\right),$$

$$b_0=\frac{a}{b}\lambda_w,$$

$$f_0^2=\frac{\pi^2 E_w I_w}{128 L_0^4 a \lambda_w},$$

$$\beta=\frac{E_w}{E_i},$$

$$\phi=\frac{D_w^2}{12\rho_i},$$

$$I_w=\frac{\pi}{64}D_w^4.$$

The collection efficiency of the wire is calculated using the method of Lozowski et al. (1983):

$$\epsilon=\begin{cases} 0.489(\log_{10}(8K_0))^{1.978} & 0.125\le K_0<0.9 \\ \dfrac{K_0}{\frac{\pi}{2}+K_0} & 0.9\le K_0, \end{cases} \tag{21}$$

$$K_0=0.125+\frac{K-0.125}{1+0.0967\,Re^{0.6367}},$$

$$K=\frac{\rho_w v d^2}{9\mu_a D},$$

$$Re=\frac{d v \rho_a}{\mu_a}.$$

Here, K_0 is the modified Langmuir inertia parameter, K is the Langmuir inertia parameter, Re is the Reynolds number, ρ_w is the density of water, d is the droplet diameter, ρ_a is the air density and μ_a is the air viscosity. In practice, the efficiency is essentially zero for $K_0 < 0.125$. D is the diameter of the cylinder collecting water, which in the case of the icing sensor would be D_w.

The air viscosity is calculated using Sutherland's Law:

$$\mu_a = \mu_0 \frac{T_0 + S}{T + S} \left(\frac{T}{T_0} \right)^{3/2}, \qquad (22)$$

where T is the air temperature, T_0 is a reference temperature, μ_0 is the viscosity at T_0, and S is Sutherland's temperature for air. Some convenient values are

$$T_0 = 273.15\,K\,,$$
$$\mu_0 = 1.716 \times 10^{-5} \frac{kg}{ms},$$
$$S = 110.4\text{K}.$$

References

Hill, G. "Analysis of Supercooled Liquid Water Measurements Using Microwave Radiometer and Vibrating Wire Devices." *J. Atmos. Ocean. Technol.* **11**, 1242-1252 (1994).