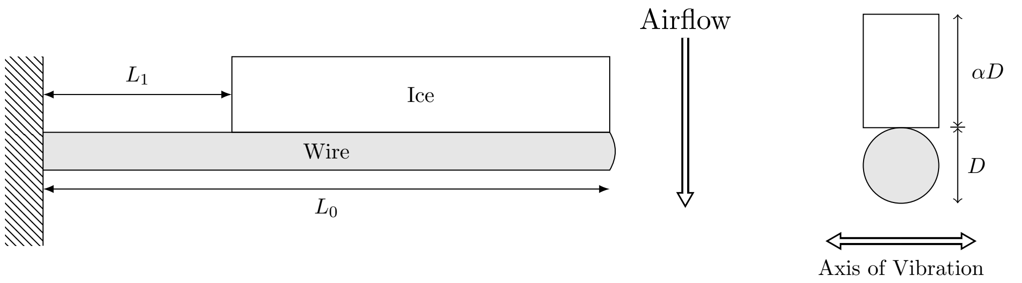
1 Introduction

Anasphere's vibrating-wire sonde records a vibrating wire's natural frequency as ice collects along its length. These frequency measurements, combined with colocated meteorological measurements, can be used to determine the supercooled liquid water content (SLWC) in the surrounding air. The following is a derivation of the equations used to calculate SLWC using the sonde's measurements. It is highly recommended that the data be smoothed before using with these equations.

2 The Wire

The system used consists of an exposed vibrating section of steel wire that accumulates ice. The wire vibrates perpendicularly to the direction of airflow. Due to boundary effects, only the forward portion of the wire collects ice. The model used for this system is shown in Figure 1. Here, the ice's height is expressed as the value α times the diameter of the wire, D.



(a) Side view (b) Head-on view

Figure 1: The wire loaded with ice

The wire's displacement equation is assumed to be

$$y = B\cos(\omega t)\left[\cos\left(\frac{\pi x}{2L_0}\right) - 1\right], \tag{1}$$

where B is an unspecified amplitude. Here, $\omega = 2\pi f$ and f is the wire's vibration frequency.

3 Frequency

Using the calculations shown in appendix A, we can find an equation for the frequency of the wire by equating the maximum potential and kinetic energies of the wire. The potential energy is given by

$$PE = \frac{E_w I_w}{2}\int_0^{L_0} y_{xx}^2 \,\mathrm{d}x + \frac{E_i I_i}{2}\int_{L_1}^{L_0} y_{xx}^2 \,\mathrm{d}x \tag{2}$$

$$= \frac{E_w I_w}{2}\left(\frac{\pi^4 B^2}{32 L_0^4}\right) L_0 \cos^2(\omega t) + \frac{E_i I_i}{2}\left(\frac{\pi^4 B^2}{32 L_0^4}\right) c L_0 \cos^2(\omega t)$$

$$PE = \left(\frac{\pi^4 B^2}{64 L_0^4}\right) E_w L_0 \cos^2(\omega t)\left(I_w + \frac{c I_i}{\beta}\right), \tag{3}$$

where $\beta = E_w/E_i$. The kinetic energy of the wire is given by

$$KE = \frac{\lambda_w}{2}\int_0^{L_0} \dot{y}^2\,\mathrm{d}x + \frac{\lambda_i}{2}\int_{L_1}^{L_0} \dot{y}^2\,\mathrm{d}x \tag{4}$$

$$= \frac{\lambda_w}{2}\omega^2 B^2 a L_0 \sin^2(\omega t) + \frac{\lambda_i}{2}\omega^2 B^2 b L_0 \sin^2(\omega t)$$

$$KE = \frac{1}{2}\omega^2 B^2 L_0 \sin^2(\omega t)(a\lambda_w + b\lambda_i). \tag{5}$$

Here, the subscripts w and i indicate the wire and ice, respectively. E, I, and λ are the Young's modulus, the area moment of inertia, and linear density of the materials, respectively.

The maximum potential and kinetic energies are then

$$PE_{\max} = \left(\frac{\pi^4 B^2}{64 L_0^4}\right) E_w L_0 \left(I_w + \frac{cI_i}{\beta}\right), \tag{6}$$

$$KE_{\max} = \frac{1}{2}\omega^2 B^2 L_0 (a\lambda_w + b\lambda_i). \tag{7}$$

Equating these, multiplying both sides by ω^2, and rearranging yields

$$\omega^2 = \frac{\omega^2 PE_{\max}}{KE_{\max}}. \tag{8}$$

This method of finding the wire's frequency is known as the Rayleigh method. Using this with 6 and 7 and recalling that $\omega = 2\pi f$, yields

$$f^2 = \frac{\pi^2 E_w (I_w + cI_i/\beta)}{128 L_0^4 (a\lambda_w + b\lambda_i)}. \tag{9}$$

4 SLWC

To find an equation for SLWC, the system is modeled as a wire sweeping through a volume of air V and collecting a percentage ϵ of the SLW contained in that volume as ice mass M_i:

$$SLWC = \frac{1}{\epsilon}\frac{\Delta M_i}{V}. \tag{10}$$

Since the wire only collects ice on the end region, $V = D(L_0 - L_1)v\Delta t$, where v is the speed of the wire and t is time. Equation 10 becomes

$$SLWC = \frac{1}{\epsilon}\frac{\Delta M_i}{Dv(L_0 - L_1)\Delta t}$$

$$= \frac{1}{\epsilon}\frac{\Delta \lambda_i}{Dv\Delta t}$$

$$= \frac{1}{\epsilon D v}\dot{\lambda}_i, \tag{11}$$

where $\dot{\lambda}_i$ is the time derivative. The percentage collected, or collection efficiency, is calculated using the method of Lozowski et al. (1983). The details of this method can be found in appendix B. Using the results from appendix C, specifically Equation 42, our final equation for SLWC is

$$SLWC = \frac{2b_0 \beta I_w f_0^2 f (b_0 c\phi - \beta I_w)}{\epsilon D v (b_0 c \phi f_0^2 - \beta I_w f^2)^2} \dot{f}, \tag{12}$$

where

$$\begin{aligned}
a &= \frac{3\pi - 8}{2\pi}, \\
b &= a - \frac{3L_1}{2L_0} - \frac{1}{2\pi} \sin\left(\frac{\pi L_1}{L_0}\right) + \frac{4}{\pi} \sin\left(\frac{\pi L_1}{2L_0}\right), \\
c &= 1 - \frac{L_1}{L_0} - \frac{1}{\pi} \sin\left(\frac{\pi L_1}{L_0}\right), \\
b_0 &= \frac{a}{b} \lambda_w, \\
f_0^2 &= \frac{\pi^2 E_w I_w}{128 L_0^4 a \lambda_w}, \\
\beta &= \frac{E_w}{E_i}, \\
\phi &= \frac{D^2}{12\rho_i}, \\
I_w &= \frac{\pi}{64} D^4.
\end{aligned}$$

5 Idealized SLWC

Equation 12 assumes that the stiffness of the ice has a significant effect on the frequency of the wire. If, instead, the ice's stiffness is considered to be negligible, then Equation 12 can be reduced to an idealized form. This is done by rearranging it and taking its limit as β approaches ∞,

$$\begin{aligned}
\lim_{\beta \to \infty} SLWC &= \frac{2b_0 I_w f_0^2 f \left(\frac{b_0 c \phi}{\beta} - I_w\right)}{\epsilon D v \left(\frac{b_0 c \phi}{\beta} f_0^2 - I_w f^2\right)^2} \dot{f} \\
&= -\frac{2 b_0 I_w^2 f_0^2 f}{\epsilon D v I_w f^4} \dot{f},
\end{aligned}$$

which gives

$$SLWC_{ideal} = -\frac{2b_0 f_0^2}{\epsilon D v f^3} \dot{f}. \tag{13}$$

Appendices

A Energy Integrals

The wire's displacement equation:

$$y = B\cos(\omega t)\left[\cos\left(\frac{\pi x}{2L_0}\right) - 1\right]. \tag{14}$$

A.1 Derivatives

$$\dot{y} = -\omega B\sin(\omega t)\left[\cos\left(\frac{\pi x}{2L_0}\right) - 1\right] \tag{15}$$

$$\dot{y}^2 = \omega^2 B^2\sin^2(\omega t)\left[\cos^2\left(\frac{\pi x}{2L_0}\right) - 2\cos\left(\frac{\pi x}{2L_0}\right) + 1\right] \tag{16}$$

$$y_x = -\left(\frac{\pi B}{2L_0}\right)\cos(\omega t)\sin\left(\frac{\pi x}{2L_0}\right) \tag{17}$$

$$y_{xx} = -\left(\frac{\pi^2 B}{4L_0^2}\right)\cos(\omega t)\cos\left(\frac{\pi x}{2L_0}\right) \tag{18}$$

$$y_{xx}^2 = \left(\frac{\pi^4 B^2}{16L_0^4}\right)\cos^2(\omega t)\cos^2\left(\frac{\pi x}{2L_0}\right) \tag{19}$$

A.2 Integrals

A.2.1 Time-Derivative Integrals

$$\begin{aligned}
\int \dot{y}^2\,\mathrm{d}x &= \omega^2 B^2\sin^2(\omega t)\int\left[\cos^2\left(\frac{\pi x}{2L_0}\right) - 2\cos\left(\frac{\pi x}{2L_0}\right) + 1\right]\mathrm{d}x \\
&= \omega^2 B^2\sin^2(\omega t)\left[\frac{1}{2}\int\left(1 + \cos\left(\frac{\pi x}{L_0}\right)\right)\mathrm{d}x - \frac{4L_0}{\pi}\sin\left(\frac{\pi x}{2L_0}\right) + x\right] \\
&= \omega^2 B^2\sin^2(\omega t)\left[\frac{3}{2}x + \frac{L_0}{2\pi}\sin\left(\frac{\pi x}{L_0}\right) - \frac{4L_0}{\pi}\sin\left(\frac{\pi x}{2L_0}\right)\right]
\end{aligned}$$

$$\begin{aligned}
\int_0^{L_0} \dot{y}^2\,\mathrm{d}x &= \omega^2 B^2\sin^2(\omega t)\left[\frac{3}{2}x + \frac{L_0}{2\pi}\sin\left(\frac{\pi x}{L_0}\right) - \frac{4L_0}{\pi}\sin\left(\frac{\pi x}{2L_0}\right)\right]\Bigg|_0^{L_0} \\
&= \omega^2 B^2\sin^2(\omega t)\left[\frac{3}{2}L_0 + \frac{L_0}{2\pi}(\sin(\pi) - \sin(0)) - \frac{4L_0}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right)\right] \\
&= \omega^2 B^2\sin^2(\omega t)\left[\frac{3}{2}L_0 - \frac{4}{\pi}L_0\right] \\
&= \omega^2 B^2 a L_0\sin^2(\omega t)
\end{aligned}$$

$$a = \frac{3}{2} - \frac{4}{\pi} = \frac{3\pi - 8}{2\pi}$$

$$
\begin{aligned}
\int_{L_1}^{L_0} \dot{y}^2 \,\mathrm{d}x &= \omega^2 B^2 \sin^2(\omega t) \left[\frac{3}{2}x + \frac{L_0}{2\pi} \sin\left(\frac{\pi x}{L_0}\right) - \frac{4L_0}{\pi} \sin\left(\frac{\pi x}{2L_0}\right) \right|_{L_1}^{L_0} \\
&= \omega^2 B^2 \sin^2(\omega t) \left[\frac{3}{2}(L_0 - L_1) + \frac{L_0}{2\pi} \left(\sin(\pi) - \sin\left(\frac{\pi L_1}{L_0}\right) \right) \right. \\
&\quad \left. - \frac{4L_0}{\pi} \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi L_1}{2L_0}\right) \right) \right] \\
&= \omega^2 B^2 \sin^2(\omega t) \left[\frac{3}{2}(L_0 - L_1) - \frac{L_0}{2\pi} \sin\left(\frac{\pi L_1}{L_0}\right) - \frac{4L_0}{\pi} + \frac{4L_0}{\pi} \sin\left(\frac{\pi L_1}{2L_0}\right) \right] \\
&= \omega^2 B^2 \sin^2(\omega t) \left[\left(\frac{3}{2} - \frac{4}{\pi}\right) L_0 - \frac{3}{2}L_1 - \frac{L_0}{2\pi} \sin\left(\frac{\pi L_1}{L_0}\right) + \frac{4L_0}{\pi} \sin\left(\frac{\pi L_1}{2L_0}\right) \right] \\
&= \omega^2 B^2 b L_0 \sin^2(\omega t)
\end{aligned}
$$

$$
b = a - \frac{3L_1}{2L_0} - \frac{1}{2\pi} \sin\left(\frac{\pi L_1}{L_0}\right) + \frac{4}{\pi} \sin\left(\frac{\pi L_1}{2L_0}\right)
$$

A.2.2 X-Derivative Integrals

$$
\begin{aligned}
\int y_{xx}^2 \,\mathrm{d}x &= \left(\frac{\pi^4 B^2}{16 L_0^4}\right) \cos^2(\omega t) \int \cos^2\left(\frac{\pi x}{2L_0}\right) \mathrm{d}x \\
&= \frac{1}{2} \left(\frac{\pi^4 B^2}{16 L_0^4}\right) \cos^2(\omega t) \int \left(1 + \cos\left(\frac{\pi x}{L_0}\right)\right) \mathrm{d}x \\
&= \left(\frac{\pi^4 B^2}{32 L_0^4}\right) \cos^2(\omega t) \left[x + \frac{L_0}{\pi} \sin\left(\frac{\pi x}{L_0}\right) \right]
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{L_0} y_{xx}^2 \,\mathrm{d}x &= \left(\frac{\pi^4 B^2}{32 L_0^4}\right) \cos^2(\omega t) \left[x + \frac{L_0}{\pi} \sin\left(\frac{\pi x}{L_0}\right) \right|_0^{L_0} \\
&= \left(\frac{\pi^4 B^2}{32 L_0^4}\right) L_0 \cos^2(\omega t)
\end{aligned}
$$

$$
\begin{aligned}
\int_{L_1}^{L_0} y_{xx}^2 \,\mathrm{d}x &= \left(\frac{\pi^4 B^2}{32 L_0^4}\right) \cos^2(\omega t) \left[x + \frac{L_0}{\pi} \sin\left(\frac{\pi x}{L_0}\right) \right|_{L_1}^{L_0} \\
&= \left(\frac{\pi^4 B^2}{32 L_0^4}\right) \cos^2(\omega t) \left[L_0 - L_1 + \frac{L_0}{\pi} \sin(\pi) - \frac{L_0}{\pi} \sin\left(\frac{\pi L_1}{L_0}\right) \right] \\
&= \left(\frac{\pi^4 B^2}{32 L_0^4}\right) L_0 \cos^2(\omega t) \left[1 - \frac{L_1}{L_0} - \frac{1}{\pi} \sin\left(\frac{\pi L_1}{L_0}\right) \right] \\
&= \left(\frac{\pi^4 B^2}{32 L_0^4}\right) c L_0 \cos^2(\omega t)
\end{aligned}
$$

$$
c = 1 - \frac{L_1}{L_0} - \frac{1}{\pi} \sin\left(\frac{\pi L_1}{L_0}\right)
$$

A.2.3 Integral Result Summary

$$\int_0^{L_0} \dot{y}^2 \,\mathrm{d}x = \omega^2 B^2 a L_0 \sin^2(\omega t) \tag{20}$$

$$\int_{L_1}^{L_0} \dot{y}^2 \,\mathrm{d}x = \omega^2 B^2 b L_0 \sin^2(\omega t) \tag{21}$$

$$\int_0^{L_0} y_{xx}^2 \,\mathrm{d}x = \left(\frac{\pi^4 B^2}{32 L_0^4}\right) L_0 \cos^2(\omega t) \tag{22}$$

$$\int_{L_1}^{L_0} y_{xx}^2 \,\mathrm{d}x = \left(\frac{\pi^4 B^2}{32 L_0^4}\right) c L_0 \cos^2(\omega t) \tag{23}$$

$$a = \frac{3}{2} - \frac{4}{\pi} = \frac{3\pi - 8}{2\pi} \tag{24}$$

$$b = a - \frac{3L_1}{2L_0} - \frac{1}{2\pi}\sin\left(\frac{\pi L_1}{L_0}\right) + \frac{4}{\pi}\sin\left(\frac{\pi L_1}{2L_0}\right) \tag{25}$$

$$c = 1 - \frac{L_1}{L_0} - \frac{1}{\pi}\sin\left(\frac{\pi L_1}{L_0}\right) \tag{26}$$

B Collection Efficiency

The collection efficiency is calculated using the method of Lozowski et al:

$$\epsilon = \begin{cases} 0.498(\log_{10}(8K_0))^{1.978} & 0.125 \leq K_0 < 0.9 \\ \dfrac{K_0}{\frac{\pi}{2} + K_0} & 0.9 \leq K_0, \end{cases} \tag{27}$$

$$K_0 = 0.125 + \frac{K - 0.125}{1 + 0.967 Re^{0.6367}}, \tag{28}$$

$$K = \frac{\rho_{water} v d^2}{9 \mu_a D}, \tag{29}$$

$$Re = \frac{d v \rho_a}{\mu_a}. \tag{30}$$

Here, K_0 is the modified Langmuir inertia parameter, K is Langmuir inertia parameter, Re is the Reynolds number, ρ_{water} is the density of water, v is the air velocity, d is the SLW droplet diameter, ρ_a is the air density, μ_a is the air viscosity and D is the wire diameter. In practice, ϵ is essentially zero for $K_0 < 0.125$.

The air viscosity is calculated using Sutherland's Law:

$$\mu_a = \mu_0 \frac{T_0 + S}{T + S}\left(\frac{T}{T_0}\right)^{3/2} \tag{31}$$

where T is the air temperature, T_0 is a reference temperature, μ_0 is the viscosity at T_0, and

S is Sutherland's temperature for air. Some convenient values are

$$T_0 = 273.15\,\text{K},$$
$$\mu_0 = 1.716 \times 10^{-5}\,\frac{\text{kg}}{\text{m s}},$$
$$S = 110.4\,\text{K}.$$

C Time-Derivative of Ice Linear Mass Density

In order to find $\dot{\lambda}_i$, we have to write I_i in Equation 9 in terms of λ_i. The ice's area moment of inertia is given by

$$I_i = \frac{1}{12}\alpha D^4. \tag{32}$$

We can eliminate α by writing it in terms of λ_i, assuming an idealized ice accretion shape:

$$\lambda_i = \rho_i A_i = \rho_i \alpha D^2 \tag{33}$$
$$\alpha = \frac{\lambda_i}{\rho_i D^2}, \tag{34}$$

where A_i is the cross sectional area of the ice and ρ_i is ice density. Using this we get

$$I_i = \frac{D^2}{12\rho_i}\lambda_i = \phi\lambda_i, \tag{35}$$

where

$$\phi = \frac{D^2}{12\rho_i}. \tag{36}$$

We can also use the wire's natural frequency, f_0, which is defined as

$$\begin{aligned} f_0^2 &= f^2(\lambda_i = 0) \\ &= \frac{\pi^2 E_w I_w}{128 L_0^4 a \lambda_w};. \end{aligned} \tag{37}$$

After rearranging,

$$\frac{\pi^2 E_w}{128 L_0^4} = \frac{a\lambda_w f_0^2}{I_w}.$$

Equation 9 then becomes

$$f^2 = \frac{a f_0^2 \lambda_w \left(I_w + \frac{c\phi}{\beta}\lambda_i\right)}{I_w(a\lambda_w + b\lambda_i)}. \tag{38}$$

We can now find λ_i:

$$\delta = \frac{c\phi}{\beta}, \tag{39}$$

$$
\begin{aligned}
f^2 I_w(a\lambda_w + b\lambda_i) &= a f_0^2 \lambda_w \left(I_w + \frac{c\phi}{\beta}\lambda_i \right) \\
a f^2 I_w \lambda_w + b f^2 I_w \lambda_i &= a f_0^2 I_w \lambda_w + a\delta f_0^2 \lambda_w \lambda_i \\
a\delta f_0^2 \lambda_w \lambda_i - b f^2 I_w \lambda_i &= a f^2 I_w \lambda_w - a f_0^2 I_w \lambda_w \\
&= a I_w \lambda_w (f^2 - f_0^2),
\end{aligned}
$$

$$
b_0 = \frac{a}{b}\lambda_w, \tag{40}
$$

$$
\begin{aligned}
(b_0 \delta f_0^2 - f^2 I_w)\lambda_i &= b_0 I_w (f^2 - f_0^2) \\
\lambda_i &= \frac{b_0 I_w (f^2 - f_0^2)}{b_0 \delta f_0^2 - I_w f^2} \\
\lambda_i &= \frac{b_0 I_w \left(\frac{f^2}{f_0^2} - 1 \right)}{b_0 \delta - \frac{f^2}{f_0^2} I_w}.
\end{aligned} \tag{41}
$$

The derivative is then:

$$
\begin{aligned}
\dot{\lambda}_i &= \frac{\left(2 b_0 I_w \frac{f}{f_0^2}\dot{f}\right)\left(b_0\delta - I_w \frac{f^2}{f_0^2}\right) - \left(b_0 I_w \left(\frac{f^2}{f_0^2} - 1\right)\right)\left(-2 I_w \frac{f}{f_0^2}\dot{f}\right)}{\left(b_0\delta - I_w \frac{f^2}{f_0^2}\right)^2} \\
&= \frac{\left(2 b_0 I_w \frac{f}{f_0^2}\dot{f}\right)\left(b_0\delta - I_w \frac{f^2}{f_0^2} + I_w \left(\frac{f^2}{f_0^2} - 1\right)\right)}{\left(b_0\delta - I_w \frac{f^2}{f_0^2}\right)^2} \\
&= \frac{2 b_0 I_w \frac{f}{f_0^2}\dot{f}(b_0\delta - I_w)}{\left(b_0\delta - I_w \frac{f^2}{f_0^2}\right)^2} \\
&= \frac{2 b_0 I_w f_0^2 f (b_0\delta - I_w)}{(b_0\delta f_0^2 - I_w f^2)^2}\dot{f}.
\end{aligned}
$$

Substituting the value for δ back in, we get

$$
\boxed{\dot{\lambda}_i = \frac{2 b_0 \beta I_w f_0^2 f (b_0 c\phi - \beta I_w)}{(b_0 c\phi f_0^2 - \beta I_w f^2)^2}\dot{f}.} \tag{42}
$$