



Supercooled Liquid Water Content (SLWC) Sensor Equations

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SLWC Equations.

The following discussion pertains to Anasphere's supercooled liquid water content (SLWC) sensor, which measures the supercooled liquid water content of clouds. Key user-provided parameters include the median volume diameter of droplets, air velocity, and ambient meteorological parameters.

A separate discussion pertains to the related liquid water content (LWC) sensor; do not use the following math for that sensor.

Very important: it is recommended that the raw frequency data be smoothed before applying the following equations. We do not recommend using a moving average. Rather, applying a Savitzky–Golay filter with a window of nominally 15 points (subject to case-by-case optimization) is recommended to eliminate higher frequency noise.

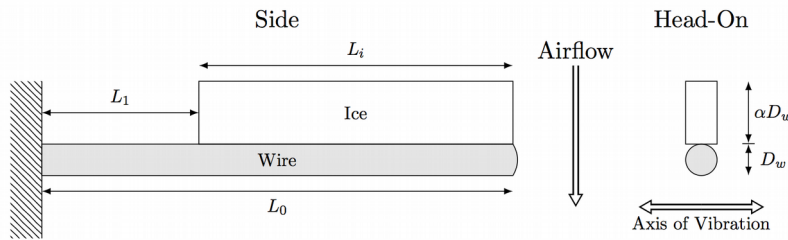


Figure 1. SLWC wire loaded with ice.

The SLWC sensor is modeled as shown in Figure 1. Here, α is a scalar allowing the ice's height to be expressed in terms of D_w . This is the same model used in Hill (1994). An equation for SLWC is found by starting with the basic equation for a length of wire sweeping through a volume V of air, with speed v , and collecting a percentage (collection efficiency) ϵ of the SLW contained in that volume as ice on its forward edge:

$$SLWC = \frac{\Delta M_i}{\epsilon V} = \frac{\Delta M_i}{\epsilon D_w L_i v \Delta t}. \quad (1)$$

M_i is the accumulated ice mass and t is time. Turning the mass into a linear density λ_i along the accumulation length and reducing the changing terms into infinitesimals, this becomes

$$\begin{aligned} SLWC &= \frac{\Delta \lambda_i}{\epsilon D_w v \Delta t}, \\ &= \frac{1}{\epsilon D_w v} \frac{d \lambda_i}{dt}, \\ &= \frac{1}{\epsilon D_w v} \dot{\lambda}_i. \end{aligned} \quad (2)$$

Next, the linear density needs to be expressed in terms of the wire's frequency f and then differentiated with respect to time. To do this, the Rayleigh method is used:

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$$\omega^2 = \frac{\omega^2 PE_{max}}{KE_{max}}, \quad (3)$$

where $\omega = 2\pi f$ and PE_{max} and KE_{max} are the maximum potential and kinetic energy of the wire, respectively. The displacement equation for the wire is assumed to be

$$y = B \cos(\omega t) \left(\cos\left(\frac{\pi x}{2L_0}\right) - 1 \right), \quad (4)$$

where B is an unspecified amplitude. The potential energy of the wire is then given by

$$\begin{aligned} PE &= \frac{E_w I_w}{2} \int_0^{L_0} y_{xx}^2 dx + \frac{E_i I_i}{2} \int_{L_1}^{L_0} y_{xx}^2 dx, \\ &= \left(\frac{\pi^4 B^2}{64 L_0^4} \right) E_w L_0 \cos^2(\omega t) \left(I_w + \frac{c I_i}{\beta} \right), \end{aligned} \quad (5)$$

where y_{xx} is the second derivative of y with respect to x , E is the Young's modulus, I is the area-moment of inertia, and $\beta = E_w/E_i$. Here, the w subscript indicates the wire and the i subscript indicates the ice. The area-moments of inertia are

$$I_w = \frac{\pi}{64} D_w^4, \quad (6)$$

$$I_i = \frac{1}{12} \alpha D_w^4. \quad (7)$$

The kinetic energy of the wire is

$$\begin{aligned} KE &= \frac{\lambda_w}{2} \int_0^{L_0} \dot{y}^2 dx + \frac{\lambda_i}{2} \int_{L_1}^{L_0} \dot{y}^2 dx, \\ &= \frac{1}{2} \omega^2 B^2 L_0 \sin^2(\omega t) (a \lambda_w + b \lambda_i), \end{aligned} \quad (8)$$

where λ_w is the linear density of the wire. The constants a , b and c are given by

$$\begin{aligned} a &= \frac{3\pi - 8}{2\pi} \\ b &= a - \frac{3L_1}{2L_0} - \frac{1}{2\pi} \sin\left(\frac{\pi L_1}{L_0}\right) + \frac{4}{\pi} \sin\left(\frac{\pi L_1}{2L_0}\right) \\ c &= 1 - \frac{L_1}{L_0} - \frac{1}{\pi} \sin\left(\frac{\pi L_1}{L_0}\right). \end{aligned} \quad (9)$$

Setting the sine and cosine terms in (5) and (8) to one and using them with (3), and recalling that $\omega = 2\pi f$, yields

$$f^2 = \frac{\pi^2 E_w \left(I_w + \frac{c I_i}{\beta} \right)}{128 L_0^4 (a \lambda_w + b \lambda_i)}. \quad (10)$$

To find $\lambda_i(f)$, I_i must first be written in terms of the ice's linear density. This is done by noting that

$$\lambda_i = \rho_i A_i = \rho_i \alpha D_w^2, \quad (11)$$

where ρ_i is the ice's density and A_i is the ice's cross-sectional area, then solving for α ,

$$\alpha = \frac{\lambda_i}{\rho_i D_w^2}, \quad (12)$$

and using the result in (7). I_i becomes

$$I_i = \frac{D_w^2}{12 \rho_i} \lambda_i = \Phi \lambda_i, \quad (13)$$

where

$$\Phi = \frac{D_w^2}{12 \rho_i}. \quad (14)$$

Thus, (10) becomes

$$f^2 = \frac{a f_0^2 \lambda_w \left(I_w + \frac{c \Phi}{\beta} \lambda_i \right)}{I_w (a \lambda_w + b \lambda_i)}, \quad (15)$$

where f_0 is the wire's fundamental frequency:

$$\begin{aligned} f_0^2 &= f^2(\lambda_i = 0) \\ &= \frac{\pi^2 E_w I_w}{128 L_0^4 a \lambda_w}. \end{aligned} \quad (16)$$

From here, through some algebra, the ice's linear density can be written as a function of the wire's frequency:

$$\lambda_i = \frac{b_0 I_w \left(\frac{f^2}{f_0^2} - 1 \right)}{\frac{b_0 c \Phi}{\beta} - \frac{f^2}{f_0^2} I_w}, \quad (17)$$

$$b_0 = \frac{a}{b} \lambda_w. \quad (18)$$

The time-derivative of the ice's linear density is then

$$\dot{\lambda}_i = \frac{2 b_0 \beta I_w f_0^2 f (b_0 c \Phi - \beta I_w)}{(b_0 c \Phi f_0^2 - \beta I_w f^2)^2} \dot{f}. \quad (19)$$

Using this in (2) produces

$$SLWC = \frac{2 b_0 \beta I_w f_0^2 f (b_0 c \Phi - \beta I_w)}{\epsilon D v (b_0 c \Phi f_0^2 - \beta I_w f^2)^2} \dot{f} \quad (20)$$

as the final equation for SLWC, where

$$\begin{aligned}
 a &= \frac{3\pi - 8}{2\pi}, \\
 b &= a - \frac{3L_1}{2L_0} - \frac{1}{2\pi} \sin\left(\frac{\pi L_1}{L_0}\right) + \frac{4}{\pi} \sin\left(\frac{\pi L_1}{2L_0}\right), \\
 c &= 1 - \frac{L_1}{L_0} - \frac{1}{\pi} \sin\left(\frac{\pi L_1}{L_0}\right), \\
 b_0 &= \frac{a}{b} \lambda_w, \\
 f_0^2 &= \frac{\pi^2 E_w I_w}{128 L_0^4 a \lambda_w}, \\
 \beta &= \frac{E_w}{E_i}, \\
 \phi &= \frac{D_w^2}{12 \rho_i}, \\
 I_w &= \frac{\pi}{64} D_w^4.
 \end{aligned}$$

The collection efficiency of the wire is calculated using the method of Lozowski et al. (1983):

$$\epsilon = \begin{cases} 0.489 (\log_{10}(8K_0))^{1.978} & 0.125 \leq K_0 < 0.9 \\ \frac{K_0}{\frac{\pi}{2} + K_0} & 0.9 \leq K_0, \end{cases} \quad (21)$$

$$K_0 = 0.125 + \frac{K - 0.125}{1 + 0.967 Re^{0.6367}},$$

$$K = \frac{\rho_w \nu d^2}{9 \mu_a D},$$

$$Re = \frac{d \nu \rho_a}{\mu_a}.$$

Here, K_0 is the modified Langmuir inertia parameter, K is the Langmuir inertia parameter, Re is the Reynolds number, ρ_w is the density of water, d is the droplet diameter, ρ_a is the air density and μ_a is the air viscosity. In practice, the efficiency is essentially zero for $K_0 < 0.125$. D is the diameter of the cylinder collecting water, which in the case of the icing sensor would be D_w .

The air viscosity is calculated using Sutherland's Law:

$$\mu_a = \mu_0 \frac{T_0 + S}{T + S} \left(\frac{T}{T_0} \right)^{3/2}, \quad (22)$$

where T is the air temperature, T_0 is a reference temperature, μ_0 is the viscosity at T_0 , and S is Sutherland's temperature for air. Some convenient values are

$$\begin{aligned} T_0 &= 273.15 \text{ K}, \\ \mu_0 &= 1.716 \times 10^{-5} \frac{\text{kg}}{\text{ms}}, \\ S &= 110.4 \text{ K}. \end{aligned}$$

References

Hill, G. "Analysis of Supercooled Liquid Water Measurements Using Microwave Radiometer and Vibrating Wire Devices." *J. Atmos. Ocean. Technol.* **11**, 1242-1252 (1994).